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Clohessy–Wiltshire Equations Modified to Include Quadratic Drag

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The relative-motion equations of a spacecraft in the vicinity of a satellite in an orbit that is not highly eccentric but decays as a result of drag are investigated. Because the initial orbit is not highly eccentric, the equations of motion of both the satellite and the spacecraft can be approximated by simpler equations. Some transformations are then applied to the equations of relative motion. If the drag is quadratic in the magnitude of the velocity and varies inversely with the distance from the center of attraction, the equations simplify further. There are some interesting consequences if the two objects in orbit have the same drag constant. In these cases, the transformed equations of relative motion generalize the Tschauner–Hempel equations and asymptotically approach an extension of the Clohessy–Wiltshire equations, modified to include the quadratic drag model. If the initial orbit is circular, the relative-motion equations become the new modified Clohessy–Wiltshire equations. The closed-form solution of these equations has similar structure to that of the Clohessy–Wiltshire equations. Among the potential applications of these new equations are fast preliminary studies for terminal rendezvous, station keeping, formation flying, and constellations of satellites.

Introduction

In a previous work,[1] we presented equations and solutions representing the terminal phase of a rendezvous of a satellite and spacecraft in the presence of atmospheric drag that is assumed to be linear in the orbital velocity. Although the assumption of drag that is linear in the velocity is not realistic, the transformations described in that paper simplified the equations of relative motion to the extent that closed-form solutions were attainable in some cases. The present paper extends that approach to a class of problems in which the drag is quadratic in the magnitude of the orbital velocity. We shall show that, in spite of the added difficulty, similar transformations also simplify this problem, and in some cases, we again find closed-form solutions.

There is an extensive body of literature on the search for representative equations and solutions in the terminal rendezvous problem.[2] For circular orbits, there are the Clohessy–Wiltshire[3] equations (also see Ref. 4). For elliptical orbits, there are the Tschauner–Hempel equations[5] (also see Ref. 6). Similar equations apply to general Keplerian orbits (see Ref. 7). All of these equations are based on a Newtonian gravitational force field, but the equations of relative motion can be approximated by linearized equations because their separation is very small when compared with their distance from the center of attraction. These equations have been generalized to describe rendezvous in a general central force field,[8] then adjusted to include drag that is linear in the velocity.[1]

Models of drag that is linear in the orbital velocity have been integrated into studies of the two-body problem by several authors.[9–13] Except for very special cases, the literature does not contain analytical solutions for drag that is quadratic in the magnitude of the velocity. Recently, the authors have found some success with this problem, and this present paper is a consequence of that work.[14]

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As in Ref. 14, we consider satellite orbits in which the magnitude of the radial velocity is much smaller than the magnitude of the tangential velocity, that is, the orbits would be elliptical of moderate or low eccentricity without the degradation of drag. For orbits of this type in which the spacecraft is near the satellite, the equations of motion can be approximated by simpler equations resulting in a problem that is amenable to analysis.

In the important special case in which the satellite and spacecraft have identical drag constants, the resulting transformed equations reduce to a minor generalization of the Tschauner–Hempel[5] equations. If the initial orbit is nearly circular, they reduce to a minor modification of the Clohessy–Wiltshire[3] equations. In this case, the closed-form solution of these new equations has the same structure as that of the Clohessy–Wiltshire solutions. A comparison of this solution with the solution of the Clohessy–Wiltshire equations provides an indication of the effect of drag on the relative motion. These equations may be useful in preliminary studies of missions involving terminal rendezvous, station keeping, formation flying, or constellations of satellites.

Derivation of the Equations

We consider the rendezvous problem between a spacecraft and a satellite in orbit in a central force field with a quadratic drag model. Our objective is to derive reduced differential equations for the motion of the spacecraft in the final stages of this maneuver similar to those that were derived earlier for the rendezvous problem with a linear drag model[1] or without drag.[7]

Restriction to Orbits That Initially Are Not Highly Eccentric but Decay Under Drag

First we consider an inertial frame. The equation of motion of the satellite in orbit is

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} - \alpha g(R)|\dot{\boldsymbol{R}}|\dot{\boldsymbol{R}} \tag{1}$$

In this equation, the first term on the right is gravitational acceleration resulting from a central force field; the second is drag acceleration. The position of the satellite from the center of attraction is denoted by the vector $\boldsymbol{R}$, its magnitude by $R = |\boldsymbol{R}| = (\boldsymbol{R}\cdot\boldsymbol{R})^{1/2}$, its velocity by $\dot{\boldsymbol{R}}$, where the dot indicates differentiation with respect to time and $|\dot{\boldsymbol{R}}| = (\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}$. The functions f and g are continuously differentiable with nonzero derivatives in the domain under consideration. The scalar α is a physical constant associated with the drag coefficient and the geometry of the satellite. We shall refer to this number as the drag constant. The function g represents atmospheric density and is assumed to depend only on altitude. Dependence on other variables such as time, latitude, and longitude are not being considered.

It is known[14] that the satellite moves in a plane containing the center of attraction. The preceding equations of motion, therefore, can be represented in polar coordinates. The radial component is R and the polar angle will be called θ. Let $\boldsymbol{e}_R$ and $\boldsymbol{e}_\theta$ be unit vectors in the radial and transverse directions, respectively. As a result of the angular momentum equation,[14] we must have $\dot\theta \geq 0$. We observe that

$$|\dot{\boldsymbol{R}}| = (\dot R^2 + R^2\dot\theta^2)^{\frac{1}{2}} = R\dot\theta[1 + (\dot R/R\dot\theta)^2]^{\frac{1}{2}} \tag{2}$$

The study is restricted to orbits that initially are not of high eccentricity but decay under drag. These are orbits in which the magnitude of the radial velocity is very small compared with the magnitude of the transverse velocity, that is, $|\dot R| \ll R\dot\theta$. For this reason $\dot\theta$ is not zero, and we can linearize the binomial expression in the right side of Eq. (2) so that Eq. (1) becomes

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} - \alpha g(R)R\dot\theta\dot{\boldsymbol{R}} \tag{3}$$

This equation of motion of the satellite can be written in terms of the transverse and radial components, respectively, as

$$R\ddot\theta + 2\dot R\dot\theta = -\alpha g(R)R^2\dot\theta^2 \tag{4}$$

$$\ddot R - R\dot\theta^2 = -f(R)R - \alpha g(R)\dot R R\dot\theta \tag{5}$$

For completeness, we briefly repeat some recent work[14] to obtain the orbit equation of the satellite.

Multiplying Eq. (4) by R, we recognize the product rule on the left side, then dividing by $R^2\dot\theta$, we obtain

$$\frac{\mathrm{d}(R^2\dot\theta)}{R^2\dot\theta} = -\alpha g(R)R\,\mathrm{d}\theta \tag{6}$$

Integrating this expression reveals

$$R^2\dot\theta = hJ(\theta) \tag{7}$$

where h is a constant of integration and

$$J(\theta) = \exp\left\{-\int \alpha g[R(\theta)]R(\theta)\,\mathrm{d}\theta\right\} \tag{8}$$

A well-known change of variable[15] from t to θ can be incorporated through Eq. (7). With this substitution, Eq. (5) can be expressed in terms of θ. Using a prime to denote differentiation with respect to θ, after some algebra we obtain the orbit equation,

$$R''/R - 2(R'/R)^2 + f(R)R^4/h^2J(\theta)^2 = 1 \tag{9}$$

Similarly the motion of a spacecraft near the satellite is governed by

$$\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{r}} = -f(|\boldsymbol{R}+\boldsymbol{r}|)(\boldsymbol{R}+\boldsymbol{r}) - \beta g(|\boldsymbol{R}+\boldsymbol{r}|)|\dot{\boldsymbol{R}}+\dot{\boldsymbol{r}}|(\dot{\boldsymbol{R}}+\dot{\boldsymbol{r}}) \tag{10}$$

where $\boldsymbol{r}$ is the vector representing the position of the spacecraft relative to the satellite and β is the drag constant associated with the spacecraft.

The spacecraft also moves in a plane containing the center of attraction. In this plane, we can also introduce polar coordinates for the vector $\boldsymbol{R}+\boldsymbol{r}$. The radial component is $|\boldsymbol{R}+\boldsymbol{r}| = [(\boldsymbol{R}+\boldsymbol{r})\cdot(\boldsymbol{R}+\boldsymbol{r})]^{1/2}$, and the polar angle we call ϕ. The equation of the spacecraft that is analogous to Eq. (3) is

$$\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{r}} = -f(|\boldsymbol{R}+\boldsymbol{r}|)(\boldsymbol{R}+\boldsymbol{r}) - \beta g(|\boldsymbol{R}+\boldsymbol{r}|)|\boldsymbol{R}+\boldsymbol{r}|\dot\phi(\dot{\boldsymbol{R}}+\dot{\boldsymbol{r}}) \tag{11}$$

Simplifying Assumptions and Transformation to a Rotating Frame

The results that we shall obtain are based on the following simplifying assumptions:

1) The functions f and g are continuously differentiable with $\mathrm{d}f/\mathrm{d}R \neq 0$ and $\mathrm{d}g/\mathrm{d}R \neq 0$ on a closed domain of R that does not contain zero.

2) The spacecraft is in the vicinity of the satellite, that is, $r \ll R$. This means that higher than first-order terms in r/R are negligible.

3) During the terminal phase of a rendezvous, $|\dot{\boldsymbol{r}}| \ll |\dot{\boldsymbol{R}}|$. A result of this and the preceding assumption is that $\dot\phi$ approaches $\dot\theta$ so that their difference can be neglected.

4) As already indicated, $|\dot R| \ll R\dot\theta$. This signifies that higher than first-order terms in $|\dot R/R\dot\theta|$ are negligible.

From the first assumption we can write

$$f(|\boldsymbol{R}+\boldsymbol{r}|) = f\left\{[(\boldsymbol{R}+\boldsymbol{r})\cdot(\boldsymbol{R}+\boldsymbol{r})]^{\frac{1}{2}}\right\} = f(R) + \frac{\mathrm{d}f}{\mathrm{d}R}(R)\left[\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R} + \frac{r^2}{2R}\right] + \mathcal{O}\left(\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R} + \frac{r^2}{2R}\right) \tag{12}$$

$$g(|\boldsymbol{R}+\boldsymbol{r}|) = g(R) + \frac{\mathrm{d}g}{\mathrm{d}R}(R)\left[\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R} + \frac{r^2}{2R}\right] + \mathcal{O}\left(\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R} + \frac{r^2}{2R}\right) \tag{13}$$

where $\lim_{x\to 0}\mathcal{O}(x)/x = 0$.

Substituting these expressions and Eq. (3) in Eq. (11), neglecting higher than first-order terms in r/R in view of assumption 2, and neglecting the difference between $\dot\phi$ and $\dot\theta$ according to assumption 3, we obtain

$$\ddot{\boldsymbol{r}} = -f(R)\boldsymbol{r} - f'(R)[(\boldsymbol{R}\cdot\boldsymbol{r})/R]\boldsymbol{R} - (\beta-\alpha)g(R)R\dot\theta\dot{\boldsymbol{R}} - \beta g(R)R\dot\theta\dot{\boldsymbol{r}} - \beta g(R)[(\boldsymbol{R}\cdot\boldsymbol{r})/R]\dot{\boldsymbol{R}}\dot\theta - \beta g'(R)(\boldsymbol{R}\cdot\boldsymbol{r})\dot{\boldsymbol{R}}\dot\theta \tag{14}$$

Here and in the following, primes over f and g denote differentiation with respect to R. In a coordinate system rotating with the radius vector of the satellite, Eq. (14) becomes

$$\begin{aligned}\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + (\dot{\boldsymbol{\Omega}}\times\boldsymbol{r}) &= -f(R)\boldsymbol{r} - f'(R)[(\boldsymbol{R}\cdot\boldsymbol{r})/R]\boldsymbol{R} - (\beta-\alpha)g(R)R\dot\theta(\dot{\boldsymbol{R}} + \boldsymbol{\Omega}\times\boldsymbol{R})\\ &\quad - \beta g(R)R\dot\theta[\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r}] - [\beta g(R)/R + \beta g'(R)]\\ &\quad \times(\boldsymbol{R}\cdot\boldsymbol{r})[\dot{\boldsymbol{R}} + \boldsymbol{\Omega}\times\boldsymbol{R}]\dot\theta\end{aligned} \tag{15}$$

where $\boldsymbol{\Omega}$ refers to the angular velocity of the rotating coordinate system and the dots over vectors indicate time derivatives with respect to this rotating coordinate system.

We now choose an orthonormal coordinate system attached to the satellite so that the x axis is transverse and opposed to the motion of the satellite, the y axis is in the direction of $\boldsymbol{R}$, and the z axis completes a right-handed system. In this frame, $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{R} = R(0, 1, 0)$, and $\boldsymbol{\Omega} = (0, 0, \omega)$, where $\omega = \dot\theta$. In this setting, Eq. (15) becomes

$$\ddot x + \beta g(R)R\omega\dot x = (\omega^2 - f(R))x + 2\omega\dot y + (\dot\omega + 2\beta g(R)R\omega^2 + \beta g'(R)R^2\omega^2)y + (\beta-\alpha)g(R)R^2\omega^2 \tag{16}$$

$$\ddot y + \beta g(R)R\omega\dot y = (\omega^2 - f(R) - f'(R)R - \beta g(R)\dot R\omega - \beta g'(R)R\dot R\omega)y - (\beta g(R))R\omega^2 + \dot\omega)x - 2\omega\dot x - (\beta-\alpha)g(R)R\dot R\omega \tag{17}$$

$$\ddot z + \beta g(R)R\omega\dot z = -f(R)z \tag{18}$$

Changing variables from t to θ in these equations, we obtain

$$\omega^2x'' + \omega\omega'x' + \beta g(R)R\omega^2x' = [\omega^2 - f(R)]x + 2\omega^2y' + [\omega\omega' + 2\beta g(R)R\omega^2 + \beta g'(R)R^2\omega^2]y + (\beta-\alpha)g(R)R^2\omega^2 \tag{19}$$

$$\begin{aligned}\omega^2y'' + \omega\omega'y' + \beta g(R)R\omega^2y' &= [\omega^2 - f(R) - f'(R)R - \beta g(R)\omega^2R' - \beta g'(R)R'R\omega^2]y\\ &\quad - [\beta g(R)R\omega^2 + \omega\omega']x - 2\omega^2x' - (\beta-\alpha)g(R)RR'\omega^2\end{aligned} \tag{20}$$

$$\omega^2z'' + \omega\omega'z' + \beta g(R)R\omega^2z' = -f(R)z \tag{21}$$

where primes on f and g denote differentiation with respect to the argument R; elsewhere there is no argument specified explicitly, and the primes represent differentiation with respect to θ.

Transformations That Simplify the Equations

We seek a transformation that will simplify Eqs. (19–21). The form of these equations suggests that it may be advantageous to multiply by an exponential. We change to new variables $\bar{x}$, $\bar{y}$, and $\bar{z}$ by the transformation

$$(x, y, z) = E(\theta)(\bar{x}, \bar{y}, \bar{z}) \tag{22}$$

where

$$E(\theta) = \omega^{\frac{1}{2}} \exp\left[-\frac{1}{2}\int \alpha g(R) R \,\mathrm{d}\theta\right] \tag{23}$$

We then divide by $\omega^2 E(\theta)$. After dropping the bars for brevity of notation, we obtain

$$x'' = [A(\theta, \omega, \omega', \omega'') - 1]x + 2y' + R[\alpha g(R) + \beta g'(R)R]y$$
$$+ (\beta - \alpha)g(R)[2Ry - (Rx)' + R^2/E(\theta)] \tag{24}$$

$$y'' = [A(\theta, \omega, \omega', \omega'') - 1]y - 2x'$$
$$- [f'(R)R/\omega^2 + \alpha g(R)R' + \beta g'(R)RR']y$$
$$- (\beta - \alpha)g(R)[(Ry)' + R'y + Rx + RR'/E(\theta)] \tag{25}$$

$$z'' = -A(\theta, \omega, \omega', \omega'')z - (Rz)'(\beta - \alpha)g(R) \tag{26}$$

where

$$A(\theta, \omega, \omega', \omega'') = -\tfrac{1}{2}(\omega''/\omega) + \tfrac{1}{4}(\omega'/\omega)^2 + f(R)/\omega^2$$
$$- \alpha g(R)R\omega'/\omega - \tfrac{1}{4}\alpha^2 g(R)^2 R^2 - \tfrac{1}{2}\alpha g(R)R' - \tfrac{1}{2}\alpha g'(R)RR' \tag{27}$$

These equations are still cumbersome, but we can substitute $\omega = hJ(\theta)/R^2$ from Eq. (7) into the right-hand side of Eq. (27), so that $A(\theta, \omega, \omega', \omega'')$ becomes the left-hand side of Eq. (9). It follows then that $A(\theta, \omega, \omega', \omega'') = 1$. Equations (24–26) reduce, therefore, to the simpler form

$$x'' = 2y' + R[\alpha g(R) + \beta g'(R)R]y$$
$$+ (\beta - \alpha)g(R)[2Ry - (Rx)' + R^2/E(\theta)] \tag{28}$$

$$y'' = -2x' - [f'(R)R/\omega^2 + \alpha g(R)R' + \beta g'(R)RR']y$$
$$- (\beta - \alpha)g(R)[(Ry)' + R'y + Rx + RR'/E(\theta)] \tag{29}$$

$$z'' = -z - (Rz)'(\beta - \alpha)g(R) \tag{30}$$

Simplifications and Solutions for Certain Cases

We now consider the case of a Newtonian gravitational field, $f(R) = \mu/R^3$, and an inverse law, $g(R) = 1/R$, to model the variation in atmospheric density. The acceleration that results from drag is, therefore, $-(\alpha/R)|\dot{\boldsymbol{R}}|\dot{\boldsymbol{R}}$. We shall provide some comments on this model after presenting the equations of relative motion and the orbit equation that result from it.

Equations of Relative Motion and Solution of the Orbit Equation

When this drag law is applied, Eq. (8) simplifies to

$$J(\theta) = e^{-\alpha\theta} \tag{31}$$

and the angular momentum Eq. (7) yields

$$\omega = hR^{-2}e^{-\alpha\theta} \tag{32}$$

These expressions can be substituted into Eq. (23) to obtain

$$E(\theta) = h^{\frac{1}{2}}R^{-1}e^{-\alpha\theta} \tag{33}$$

Using the Newtonian gravitational field and the preceding expressions, Eqs. (28–30) take the form

$$x'' = 2y' + (\beta - \alpha)\left[y - x' - (R'/R)x + h^{-\frac{1}{2}}R^2 e^{\alpha\theta}\right] \tag{34}$$

$$y'' = -2x' + 3\mu h^{-2}Re^{2\alpha\theta}y$$
$$- (\beta - \alpha)\left[x + y' + (R'/R)y + h^{-\frac{1}{2}}RR'e^{\alpha\theta}\right] \tag{35}$$

$$z'' = -z - (\beta - \alpha)[z' + (R'/R)z] \tag{36}$$

For this case, a closed-form solution has recently been found for the orbit equation (9). We briefly outline this solution from Ref. 14. Observe that, if $\alpha = \beta = 0$, Eqs. (34–36) reduce to the Tschauner–Hempel[5] equations (see also Refs. 6 and 7).

When Eq. (31) is substituted into Eq. (9), the orbit equation becomes

$$R''/R - 2(R'/R)^2 + (\mu/h^2)Re^{2\alpha\theta} = 1 \tag{37}$$

Performing the change of variable $R = 1/u$, this equation simplifies to

$$u'' + u = (\mu/h^2)e^{2\alpha\theta} \tag{38}$$

The general solution of this simple differential equation is easily obtained in terms of two arbitrary constants ϵ and θ_0. Expressing this solution in terms of the variable R, we obtain the following expression for the degradation of an orbit that was initially elliptical and not highly eccentric:

$$R = \frac{h^2(1 + 4\alpha^2)/\mu}{e^{2\alpha\theta} + \epsilon\cos(\theta - \theta_0)} \tag{39}$$

The expression $\epsilon e^{-2\alpha\theta}$ may be visualized as an instantaneous eccentricity, that is, the eccentricity of the fictitious elliptical orbit that would result if the drag could be instantly removed. The function R and its derivative can be substituted into Eqs. (34–36) yielding a set of differential equations that describe the motion of the terminal phase of the rendezvous in the presence of drag. Although the coefficients are elaborate, this set of differential equations is linear because R and R' are functions of θ.

Comments on the Model

The reason for the drag model presented is the relatively simple closed-form solutions that can be obtained from it as presented here. These are especially interesting in view of the void in the literature of closed-form solutions for drag models that vary with the square of the magnitude of the velocity.

A better approximation would have the atmospheric density decrease exponentially with R. Better yet would be a curve fit based on standard atmospheric data. Unfortunately, the more realistic models have not yet resulted in closed-form solutions.

The difficulties in the model presented herein are not as restrictive as might appear at the onset. Inaccuracies caused by a poor representation of the atmospheric density increase with the range of R. One can compensate for this error by dividing this range into several smaller subintervals and reinitializing the drag constant α with each subinterval. In fact, the drag coefficient should not be considered constant over a wide range anyway, and so the partitioning into subintervals is needed also for the assumption that α is constant.

The model should be highly accurate for circular and near-circular orbits that are very high, so that the number α is nearly zero. The deterioration in altitude will be correspondingly small, as observed from Eq. (39) when α is near zero and ϵ is not large. Large values of α or ϵ are not compatible with assumption 4 of the preceding section. Future numerical studies to determine the range of initial altitudes, the values of ϵ, and the optimum size of the subintervals for the level of accuracy needed would be highly beneficial and are recommended.

Simplification for an Initially Circular Orbit

The equations simplify if the initial conditions prescribe an orbit that would be circular without drag. For an initially circular orbit, $\epsilon = 0$, and Eq. (39) becomes

$$R = R_0 e^{-2\alpha\theta} \tag{40}$$

where

$$R_0 = h^2(1 + 4\alpha^2)/\mu \tag{41}$$

In this case, Eqs. (34–36) take the more manageable form,

$$x'' = 2y' + (\beta - \alpha)\left(y - x' + 2\alpha x + h^{\frac{1}{2}} R_0^2 e^{-3\alpha\theta}\right) \tag{42}$$

$$y'' = -2x' + 3(1 + 4\alpha^2)y - (\beta - \alpha)\left(x + y' - 2\alpha y - 2\alpha h^{-\frac{1}{2}} R_0^2 e^{-3\alpha\theta}\right) \tag{43}$$

$$z'' = -z - (\beta - \alpha)(z' - 2\alpha z) \tag{44}$$

These linear nonhomogeneous differential equations with constant coefficients can be solved in closed form for specific values of α and β. Because the third equation decouples, we need only the characteristic equation for the planar motion. The characteristic polynomial $p(\lambda)$ of the resulting coefficient matrix is

$$p(\lambda) = \lambda^4 + 2\gamma\lambda^3 + (1 + \gamma^2 - 2\alpha\gamma - 12\alpha^2)\lambda^2 + \gamma(1 - 4\alpha\gamma - 12\alpha^2)\lambda + \gamma[\gamma(1 + 4\alpha^2) + 6\alpha(1 + 4\alpha^2)] \tag{45}$$

where

$$\gamma = \beta - \alpha \tag{46}$$

The roots of this characteristic equation are

$$\lambda_{1,2} = -\tfrac{1}{2}\left(\gamma \pm \sqrt{c + 2b}\right) \tag{47}$$

$$\lambda_{3,4} = -\tfrac{1}{2}\left(\gamma \pm \sqrt{c - 2b}\right) \tag{48}$$

where

$$b = (-4\gamma^2 - 32\alpha\gamma + 144\alpha^4 - 24\alpha^2 + 1)^{\frac{1}{2}} \tag{49}$$

$$c = \gamma^2 + 8\alpha\gamma + 24\alpha^2 - 2 \tag{50}$$

For given drag constants α and β, the roots (47) and (48) are easily calculated. The solution of the differential equations (42) and (43) is then straightforward.

Simplification for Identical Drag Constants

In many applications, it is desirable for the spacecraft and the satellite to remain in close proximity for long time intervals without thrust or with very limited thrust. For this reason it may be advantageous for the two objects to have identical geometry or at least identical drag constants. Setting $\beta = \alpha$ and using Eq. (39) for R, Eqs. (34–36) simplify:

$$x'' = 2y' \tag{51}$$

$$y'' = -2x' + \frac{3(1 + 4\alpha^2)y}{1 + \epsilon e^{-2\alpha\theta}\cos(\theta - \theta_0)} \tag{52}$$

$$z'' = -z \tag{53}$$

It is observed again that these equations reduce to the Tschauner–Hempel[5] equations (also see Refs. 6 and 7) if $\alpha = 0$. The Tschauner–Hempel[5] equations apply even to highly eccentric orbits. We can integrate Eq. (51) once and substitute in Eq. (52). The system reduces to one essential equation:

$$y'' = \left[\frac{3(1 + 4\alpha^2)e^{2\alpha\theta}}{e^{2\alpha\theta} + \epsilon\cos(\theta - \theta_0)} - 4\right]y + c_3 \tag{54}$$

where c_3 is a constant.

Modification of Clohessy–Wiltshire[3] Equations to Accommodate Drag

If the orbit is initially circular, then $\epsilon = 0$ in Eq. (39) and R decays exponentially as a result of drag:

$$R = (h^2/\mu)(1 + 4\alpha^2)e^{-2\alpha\theta} \tag{55}$$

Again, assuming identical drag constants, the relative-motion equations (51–53) become

$$x'' = 2y' \tag{56}$$

$$y'' = -2x' + 3(1 + 4\alpha^2)y \tag{57}$$

$$z'' = -z \tag{58}$$

Even if the orbit is not initially circular, we observe that $e^{-2\alpha\theta}$ approaches zero in the limit, so that Eqs. (51–53) approach Eqs. (56–58) as a limiting case. After sufficient time has elapsed, the motion could be studied through Eqs. (56–58).

These new equations are remarkably similar to the Clohessy–Wiltshire[3] equations (also see Ref. 4). The only difference is the addition of the term $12\alpha^2 y$ in Eq. (57) to accommodate the effect of quadratic drag. This demonstrates that the Clohessy–Wiltshire[3] equations can be modified to include the effect of drag.

The complete solution of these new equations is obtained in the same manner as for the Clohessy–Wiltshire[3] equations. Setting $\epsilon = 0$ in Eq. (54) yields

$$y'' = -\kappa^2 y + c_3 \tag{59}$$

where

$$\kappa = \sqrt{1 - 12\alpha^2} \tag{60}$$

We consider only the case of periodic motion where $\alpha < \sqrt{3}/6$, consequently $\kappa^2 > 0$. Solutions for $\alpha = \sqrt{3}/6$ and $\alpha > \sqrt{3}/6$ are straightforward but may not produce a very accurate model. (See the preceding comments on the model.) The out-of-plane motion satisfying Eq. (58) is sinusoidal. Integrating Eq. (59), one can find the complete solution of Eqs. (56) and (57) in terms of the arbitrary constants c_1, c_2, c_3, and c_4:

$$x = (2c_1/\kappa)\sin\kappa\theta - (2c_2/\kappa)\cos\kappa\theta + \left(2/\kappa^2 - \tfrac{1}{2}\right)c_3\theta + c_4 \tag{61}$$

$$x' = 2c_1\cos\kappa\theta + 2c_2\sin\kappa\theta + \left(2/\kappa^2 - \tfrac{1}{2}\right)c_3 \tag{62}$$

$$y = c_1\cos\kappa\theta + c_2\sin\kappa\theta + c_3\big/\kappa^2 \tag{63}$$

$$y' = -\kappa c_1\sin\kappa\theta + \kappa c_2\cos\kappa\theta \tag{64}$$

We observe that, when $\alpha = 0$, then $\kappa = 1$, and Eqs. (61–64) reduce to the solution of the Clohessy–Wiltshire[3] equations.

Recall that the new variables given by Eqs. (61) and (63) are transformed. The actual relative position is determined from multiplying by $E(\theta)$ according to Eqs. (22) and (23). It follows from Eqs. (33) and (55) that $E(\theta) = \mu/h^{3/2}(1 + 4\alpha^2)$, which is a constant. Equations (61–64) should be multiplied by this constant to have their correct scale. This is unnecessary, however, because the arbitrary constants c_1, c_2, c_3, and c_4 can absorb this scale factor. Similarly arbitrary constants c_5 and c_6 that arise from the sinusoidal solution of Eq. (58) can absorb it. For this reason, Eqs. (61–64) and all of the subsequent equations that follow from them refer to the actual untransformed relative positions and velocities as functions of θ.

The constants can be evaluated in terms of the initial conditions $x(0) = x_0$, $x'(0) = v_{10}$, and $y(0) = y_0$, $y'(0) = v_{20}$ as follows:

$$c_1 = y_0 - 2(2y_0 - v_{10})\big/\kappa^2 \tag{65}$$

$$c_2 = v_{20}/\kappa \tag{66}$$

$$c_3 = 2(2y_0 - v_{10}) \tag{67}$$

$$c_4 = x_0 + 2v_{20}\big/\kappa^2 \tag{68}$$

As is the planar solution of the Clohessy–Wiltshire[3] equations (see Ref. 16) the solution of the new equations is cycloidal. Equations (61) and (63) can be put in the form

$$x = (2\rho/\kappa)\sin(\kappa\theta + \psi) + \left(2/\kappa^2 - \tfrac{1}{2}\right)c_3\theta + c_4 \tag{69}$$

$$y = \rho\cos(\kappa\theta + \psi) + c_3\big/\kappa^2 \tag{70}$$

where $c_1 = \rho\cos\psi$ and $c_2 = \rho\sin\psi$. This curve is prolate, that is, it has a loop, only if $\rho > (1/\kappa - \kappa/4)|c_3| > 0$. It can be seen from Eqs. (65–68) that a loop is not possible if the initial velocities v_{10} and v_{20} are zero. We observe that Eqs. (69) and (70) define an ellipse if $v_{10} = 2y_0$. This is well known for the Clohessy–Wiltshire[3] solutions, but these new solutions differ in that the drag increases the major axis of this elliptic relative motion. With the inclusion of drag, the major axis is $2\rho/\kappa$ and the minor axis is ρ.

To illustrate the effect of drag on the relative position of the spacecraft, we consider the case where the initial relative velocity is zero, that is, $v_{10} = v_{20} = 0$. In this case, Eqs. (69) and (70) become

$$x = 2y_0(1 - 4/\kappa^2)(\sin\kappa\theta/\kappa - \theta) + x_0 \tag{71}$$

$$y = y_0(1 - 4/\kappa^2)\cos\kappa\theta + 4y_0\big/\kappa^2 \tag{72}$$

Increasing α decreases κ through Eq. (60), increasing the amplitude and the peak value of y. If $y_0 > 0$, the maximum value of y is given by

$$y_m = (8/\kappa^2 - 1)y_0 \tag{73}$$

Equations (71) and (72) are graphed in Figs. 1–3 for $x_0 = 0$, $y_0 = 1$, and $-4\pi \le \theta \le 4\pi$. Figure 1 shows the Clohessy–Wiltshire[3] solution, where $\alpha = 0$, resulting in $\kappa = 1$. In Fig. 2, $\alpha = \sqrt{3}/10 \approx 0.1732$,

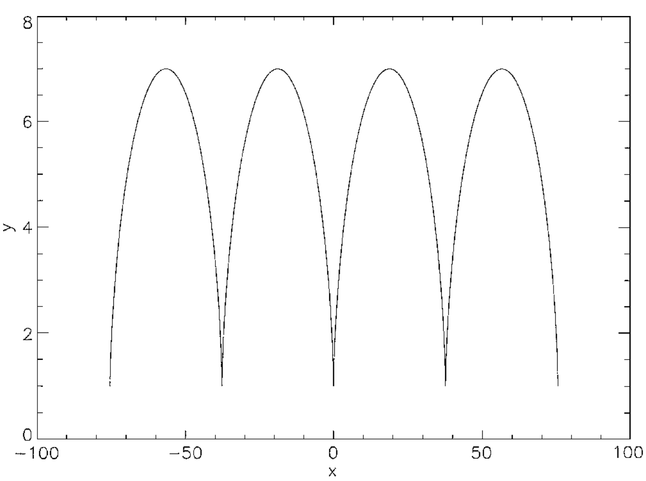

Fig. 1 Relative position of spacecraft for $\kappa = 1$ (no drag).

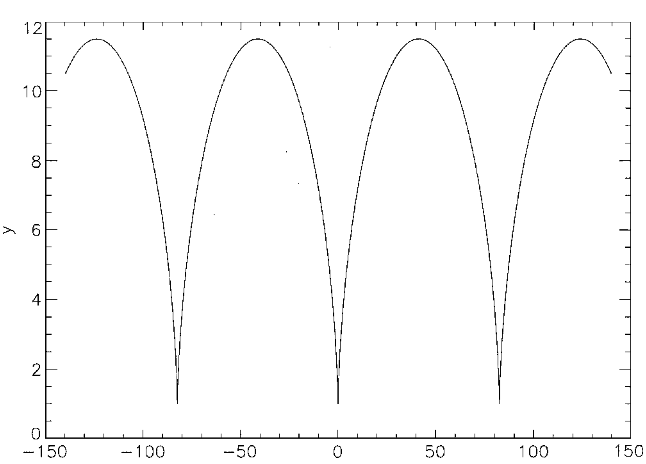

Fig. 2 Relative position of spacecraft for $\kappa = 0.8$.

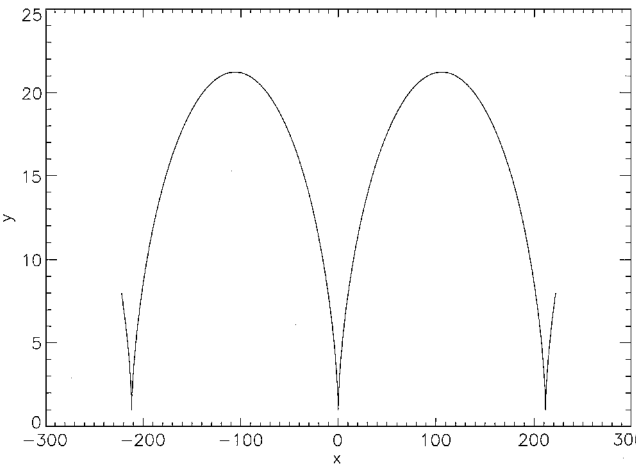

Fig. 3 Relative position of spacecraft for $\kappa = 0.6$.

resulting in $\kappa = 0.8$, and the effect of drag is seen in increasing the amplitude, raising the peaks, and increasing the period. In Fig. 3, $\alpha = 2\sqrt{3}/15 \approx 0.2309$, resulting in $\kappa = 0.6$, and the effects are more pronounced.

Flying Together

There are applications in which it is desirable for a spacecraft to remain in the vicinity of a satellite without thrusting or with a minimum of thrusting. Equations (61–64) show that the spacecraft and satellite will drift apart as a result of the secular term unless $c_3 = 0$. If one sets the initial conditions so that $v_{10} = 2y_0$, then Eq. (67) shows that the secular term is removed, and the spacecraft rotates in an elliptical orbit about the satellite. Equations (61) and (63) become

$$x = (2y_0/\kappa)\sin\kappa\theta - \left(2v_{20}\big/\kappa^2\right)\cos\kappa\theta + \left(x_0 + 2v_{20}\big/\kappa^2\right) \tag{74}$$

$$y = y_0\cos\kappa\theta + (v_{20}/\kappa)\sin\kappa\theta \tag{75}$$

The minor axis of the ellipse is $\rho = \sqrt{(y_0^2 + v_{20}^2/\kappa^2)}$. The major axis is $2\rho/\kappa$, more than twice the minor axis. Increasing the drag causes an increase in the size and eccentricity of this ellipse.

The only way that the spacecraft can remain stationary relative to the satellite is for the additional initial conditions y_0 and v_{20} to be zero. This gives the stationary solution

$$x(\theta) = x_0, \qquad y(\theta) = 0 \tag{76}$$

This result is the same as for the Clohessy–Wiltshire[3] solution. There may be applications where this solution is desirable.

State-Transition Matrix

As a final result, we construct a state-transition matrix based on the solution of the new modification of the Clohessy–Wiltshire[3] equations.

We shall denote the state vector associated with Eqs. (56–58) by $\hat{\boldsymbol{x}}(\theta) = [x(\theta), x'(\theta), y(\theta), y'(\theta), z(\theta), z'(\theta)]^T$ and the related constant vector by $\hat{\boldsymbol{c}} = (c_1, c_2, c_3, c_4, c_5, c_6)^T$. The state vector can be expressed as

$$\hat{\boldsymbol{x}}(\theta) = \Phi(\theta)\hat{\boldsymbol{c}} \tag{77}$$

where $\Phi(\theta)$ is a fundamental matrix solution associated with Eqs. (56–58). In view of Eqs. (61–64), we see that

$$\Phi(\theta) = \begin{bmatrix} 2\sin\kappa\theta/\kappa & -2\cos\kappa\theta/\kappa & \left(2/\kappa^2 - \frac{1}{2}\right)\theta & 1 & 0 & 0 \\ 2\cos\kappa\theta & 2\sin\kappa\theta & \left(2/\kappa^2 - \frac{1}{2}\right) & 0 & 0 & 0 \\ \cos\kappa\theta & \sin\kappa\theta & 1/\kappa^2 & 0 & 0 & 0 \\ -\kappa\sin\kappa\theta & \kappa\cos\kappa\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{78}$$

A fundamental matrix solution is invertible, and so, evaluating Eq. (77) at a different point θ_0 and solving for $\hat{\boldsymbol{c}}$, one obtains

$$\hat{\boldsymbol{x}}(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}\hat{\boldsymbol{x}}(\theta_0) \tag{79}$$

The matrix $M(\theta, \theta_0) = \Phi(\theta)\Phi(\theta_0)^{-1}$ is the state-transition matrix. Multiplication of a state vector $\hat{\boldsymbol{x}}(\theta_0)$ by this matrix determines the state $\hat{\boldsymbol{x}}(\theta)$. Whenever the coefficients are constant in the state equations, as is the case in Eqs. (56–58), it is known that $M(\theta, \theta_0) = M(\theta - \theta_0, 0) = \Phi(\theta - \theta_0)\Phi(0)^{-1}$. It follows from Eqs. (65–68) that

$$\Phi(0)^{-1} = \begin{bmatrix} 0 & 2/\kappa^2 & 1 - 4/\kappa^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\kappa & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2/\kappa^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{80}$$

For brevity, we set $\tau = \theta - \theta_0$. Performing the multiplication $\Phi(\tau)\Phi(0)^{-1}$, we obtain the state-transition matrix

$$M(\tau, 0) = \begin{bmatrix} 1 & 4\sin\kappa\tau/\kappa^2 + (1 - 4/\kappa^2)\tau & 2(1 - 4/\kappa^2)(\sin\kappa\tau/\kappa - \tau) & 2(1 - \cos\kappa\tau)/\kappa^2 & 0 & 0 \\ 0 & 4(\cos\kappa\tau - 1)/\kappa^2 + 1 & 2(1 - 4/\kappa^2)(\cos\kappa\tau - 1) & 2\sin\kappa\tau/\kappa & 0 & 0 \\ 0 & 2(\cos\kappa\tau - 1)/\kappa^2 & (1 - 4/\kappa^2)\cos\kappa\tau + 4/\kappa^2 & \sin\kappa\tau/\kappa & 0 & 0 \\ 0 & -2\sin\kappa\tau/\kappa & -\kappa(1 - 4/\kappa^2)\sin\kappa\tau & \cos\kappa\tau & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\tau & \sin\tau \\ 0 & 0 & 0 & 0 & -\sin\tau & \cos\tau \end{bmatrix} \tag{81}$$

Multiplication of the vector $\hat{\boldsymbol{x}}(\theta_0)$ by this matrix produces the vector $\hat{\boldsymbol{x}}(\theta)$. As a special case, one may set $\kappa = 1$ and obtain the state-transition matrix for the Clohessy–Wiltshire[3] equations.

Conclusions

We have shown that under certain reasonable assumptions the relative motion equations of a satellite and a spacecraft whose orbits decay under a quadratic drag model can be simplified if the initial orbit of the satellite is elliptical and not highly eccentric. This simplification is accomplished by the application of a successive set of transformations similar to those used in a previous work that assumed a linear drag model. If the drag constants of the spacecraft and satellite are identical, as is the case in some formation-flying applications, some remarkable simplifications occur, generalizing the Tschauner–Hempel[5] equations and the Clohessy–Wiltshire[3] equations. The generalization of the Clohessy–Wiltshire equations is very similar in form and in structure of the solution to the original. The simplicity of the solution of these new equations allows one to assess quickly the effects of drag on the relative position and velocity of a spacecraft in preliminary studies. The increase in the amplitude and period of the relative motion is evident from this solution. Among the potential applications of these new equations are fast preliminary studies for terminal rendezvous, station keeping, formation flying, and constellations of satellites.

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